Low Peclet Number Heat Transfer for Power Law Non-Newtonian Fluid With Heat Generation

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Synopsis

Heat transfer for power law non-Newtonian fluid with heat generation and axial conduction is analyzed. Radial and axial temperature distribution and the Nusselt number inside a tube are obtained in terms of nonorthogonal series expansion. Eigenvalues and eigenfunctions are given for different values of various parameters. The effects of Peclet number, power law model index, viscous dissipation, and heat generation on the temperature distribution and Nusselt number are discussed. Comparison of the present results for extreme cases with those obtained by previous workers shows good agreement.

INTRODUCTION

The study of heat transfer in non-Newtonian pipe flow has been quite extensive in the past 25 years for practical reasons. Transfer of heat to flowing polymer solutions or melts has been important in polymer processing, and simultaneous internal heat generation and transfer of thorium oxide slurries in both core and blanket regions of nuclear reactors are also of particular interest. Complete understanding of heat effect is extremely important in the preparation of good polymer concrete.

Among the numerous investigators of non-Newtonian heat transfer in laminar pipe flow, Pigford[10] examined heat transfer with Leveque's approximate solution. Lyche and Bird[8] extended the Graetz–Nusselt problem to power law fluid non-Newtonian pipe flow and obtained a semianalytical solution. Toor[12] investigated problems similar to those studied by Lyche and Bird[8] but took into account internal heat generation. Toor[12] lumped both the compression work and the heat source terms together in his analysis, and therefore the individual effects of these two terms are not easily distinguished. A recent analysis of pseudoplastic fluids with arbitrary wall heat flux has been performed by Faghri and Welty.[4] Previous workers usually neglected the axial heat conduction term in their analysis, arguing that it is generally small compared with the axial convection term. This assumption cannot be valid if high thermal diffusivity fluid flows at a low mean velocity, in which case it may be necessary to include axial conduction in the analysis. However, inclusion of the axial heat conduction term in the energy equation changes the partial differential equation from parabolic to elliptic, with the result that the eigenfunctions derived from the Graetz–Nusselt technique are no longer orthogonal. This presents some difficulty in obtaining a solution. Methods used previously for Newtonian fluid heat or mass transfer with axial conduction or diffusion[2,6] will be applied here to the non-Newtonian case.

The objective of this paper is to present a solution and to investigate convective heat transfer with axial conduction of a non-Newtonian power law fluid flowing

Journal of Applied Polymer Science, Vol. 23, 3077–3103 (1979)
 0021-8995/79/0023-3077$01.00

in a tube with internal heat generation due to compression work and the heat source.

ANALYSIS OF THE PROBLEM

When a non-Newtonian fluid is flowing in a horizontal tube with constant physical properties, several assumptions can be made in formulating the problem. One can consider a steady, hydrodynamically developed laminar flow with a constant wall temperature. Heat production inside the tube due to viscous dissipation and heat generation is taken into account. The importance of the first term can be seen in the extrusion of molten plastics, while the significance of the second term can be seen in the nuclear core or blanket regions of thorium oxide. With these assumptions, one can write the energy equation as

$$\rho C_p \frac{3s+1}{2(s+1)} u_{\text{av}} \left[1 - \left(\frac{r}{a}\right)^{(s+1)/s}\right] \frac{\partial T}{\partial x} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2}\right] + m\left|\frac{\partial U_x}{\partial r}\right|^{s-1}\left(\frac{dU_x}{dr}\right) + q \quad (1)$$

where the last two terms represent heat production due to viscous dissipation and heat generation. Note that if $s = 1$, the fluid is Newtonian; if $s < 1$, the fluid is pseudoplastic; and if $s > 1$, the fluid becomes dilatant.

The boundary conditions are

$$T(0,r) = T_0 \quad (2)$$

$$T(x,a) = T_w \quad (3)$$

$$\frac{\partial T(x,0)}{\partial r} = 0 \quad (4)$$

The inlet boundary condition (2) is used here for first approximation. A rigorous treatment can be followed by using the method of Hsu[7] and Dang.[3]

Introducing the dimensionless parameters

$$\theta = \frac{T - T_w}{T_0 - T_w}$$

$$\eta = \frac{r}{a}$$

$$\xi = \frac{2(s+1)kx}{(3s+1)\rho C_p a^2 u_{\text{av}}} = \frac{2(s+1)x}{(3s+1)aPe}$$

$$\delta = \frac{qa^2}{k(T_0 - T_w)}$$

$$\beta = \left(\frac{3s+1}{2s}\right)^{s+1} \frac{m u_{av}{}^{s+1} a^{1-s}}{k(T_0 - T_w)} = \left(\frac{3s+1}{2s}\right)^{s+1} Br$$

one can transform eqs. (1) to (4) into the following:

$$(1 - \eta^{(s+1)/s}) \frac{\partial \theta}{\partial \xi} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta}{\partial \eta}\right) + \frac{4(s+1)^2}{(3s+1)^2(Pe)^2}\frac{\partial^2 \theta}{\partial \xi^2} + \beta \eta^{(s+1)/s} + \delta \quad (5)$$

$$\theta(0,\eta) = 1 \quad (6)$$

$$\theta(\xi,1) = 0 \quad (7)$$

$$\frac{\partial\theta(\xi,0)}{\partial\eta} = 0 \tag{8}$$

One can take the solution of this set of equations as the sum of two parts $\theta_1(\eta)$ and $\theta_2(\xi,\eta)$:

$$\theta(\xi,\eta) = \theta_1(\eta) + \theta_2(\xi,\eta) \tag{9}$$

where $\theta_1(\eta)$ is the asymptotic solution obtained for the downstream region where the temperature profile is fully developed and $\theta_2(\xi,\eta)$ is the entrance region solution.

It is quite straightforward to obtain the solution for $\theta_1(\eta)$:

$$\theta_1(\eta) = \frac{\beta s^2}{(2s+1)(3s+1)}(1-\eta^{(3s+1)/s}) + \frac{\delta}{2}(1-\eta^2) \tag{10}$$

$\theta_2(\xi,\eta)$ satisfies the partial differential equation of the first three terms of eq. (5) and its corresponding suitable boundary conditions (6) to (8). Taking $\theta_2(\xi,\eta)$ in the form

$$\theta_2(\xi,\eta) = \sum_{n=1}^{\infty} A_n R_n(\eta) \exp(-\alpha_n^2 \xi) \tag{11}$$

one can obtain a set of differential equations defining $R_n(\eta)$ as

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dR_n}{d\eta}\right) + \alpha_n^2\left[1 - \eta^{(s+1)/s} + \frac{4\alpha_n^2}{(Pe)^2}\left(\frac{s+1}{3s+1}\right)\right]R_n = 0 \tag{12}$$

$$R_n(1) = 0 \tag{13}$$

$$\frac{dR_n(0)}{d\eta} = 0 \tag{14}$$

$$\sum_{n=1}^{\infty} A_n R_n(\eta) = 1 - \beta s^2/(3s+1)(2s+1)\,(1-\eta^{(3s+1)/s}) - \frac{\delta}{2}(1-\eta^2) \tag{15}$$

Equations (12) to (14) can be solved readily by either the Runge–Kutta method or in terms of a confluent hypergeometric function. The expansion coefficients A_n cannot be determined by classical methods of orthogonal expansion but can be determined by the solution of the following set of linear algebraic equations which are obtained by truncating the series to $n = 20$ in the present case instead of taking infinite terms as in eq. (11). Hence, the matrix equations that govern A_n are

$$A_n J_n + \sum_{\substack{m=1\\ m\neq n}}^{\infty} \Gamma_{n,m} A_m = I_n \tag{16}$$

where

$$I_n = \int_0^1 \left[1 - \frac{\beta s^2}{(3s+1)(2s+1)}(1-\eta^{(s+1)/s}) - \frac{\delta}{2}(1-\eta^2)\right] \times \left[1 - \eta^{(s+1)/s} + \frac{8\alpha_n^2}{(Pe)^2}\left(\frac{s+1}{3s+1}\right)^2\right]\eta R_n\,d\eta \tag{17}$$

$$J_n = \int_0^1 \left[1 - \eta^{(s+1)/s} + \frac{8\alpha_n^2}{(Pe)^2}\left(\frac{s+1}{3s+1}\right)^2\right]\eta R_n^2 d\eta \tag{18}$$

$$\Gamma_{n,m} = (\alpha_n^2 - \alpha_m^2)\frac{4}{(Pe)^2}\left(\frac{s+1}{3s+1}\right)^2 \int_0^1 \eta R_n R_m \, d\eta \tag{19}$$

Equation (16) is solved numerically by the Gauss–Seidel method after I_n, J_n, and $\Gamma_{n,m}$ are evaluated.

RESULTS AND DISCUSSION

Equations (12) to (14) are solved numerically by the iteration technique. Eigenvalues and eigenfunctions are obtained when the absolute value of $R_n(1)$ is less than 10^{-10}. In the present study, eigenvalues and their related constants are obtained for the following: $Pe = 1, 5, 10$; $s = 0.25, 0.5, 0.75, 1, 1.5, 2.0$; $\beta = -1, -0.5, 0, 0.5, 1.0$; $\delta = -1, 1$. However, only some of the eigenvalues and related constants of these parameters are reported in Tables I through XV. The results are believed to be very accurate. Comparison can be made of some simpler problems like those presented here with those available in the literature. When $Pe \to \infty$, $\beta = \delta = 0$, the present problem reduces to that solved by Lyche and Bird[8]; and when $s = 1$, $Pe \to \infty$, $\beta = \delta = 0$, this becomes the classical Graetz–Nusselt[5,9] problem. In Table XVI, the eigenvalues from the present results are compared with those of Lyche and Bird,[8] Sellars et al.,[11] and Abramowtiz.[1] Lyche and Bird[8] reported only four eigenvalues for $s = 1, 0.5$, and three for $s = 1/3$. Sellars et al.[11] reported ten approximate eigenvalues for the Graetz–Nusselt problem, while Abramowitz[1] gave five accurate eigenvalues. The results presented in this report are in good agreement with those of the previous workers. Although 20 eigenfunctions are evaluated in the present analysis, only the first three are compared with those of Lyche and Bird because these are the only values reported by them. Again (Table XVII), good agreement is obtained between the two sets of values.

Figures 1 and 2 show the first four eigenfunctions for different values of power law index s. For clarification purposes, the eigenfunction for $s = 0.5$ is omitted in these figures. It should be noted that for the present problem eigenfunctions are determined as long as Peclet number Pe and power law index s are specified, and they will be the same for all values of β and δ. The higher the order of eigenfunctions, the higher the number of eigenfunctions that cross over the zero value. The first four eigenfunctions can be used to approximate temperature distribution as well as the Nusselt number, as will be discussed later. However, if high accuracy is required, 20 or more eigenfunctions are needed for the calculations. Defining the dimensionless average temperature as

$$\theta_b(\xi) = \frac{\int_0^a u_x \theta r \, dr}{\int_0^a u_x r \, dr}$$

one can then obtain an expression for the dimensionless average temperature by substituting eqs. (9) to (11) into the above expression and evaluating

$$\theta_b(\xi) = \frac{\beta s^2(23s^4 + 35s^3 + 28s^2 + 9s + 1)}{(2s+1)(5s+1)(3s+1)(s+1)(8s^2+5s+1)} + \frac{\delta(7s+1)}{4(5s+1)} + \frac{2(3s+1)}{s+1}\sum_{n=1}^{\infty} A_n \exp(-\alpha_n^2 \xi) \int_0^1 (1 - \eta^{(s+1)/s}) R_n(\eta) \eta \, d\eta \tag{20}$$

TABLE I
Eigenvalues and Related Constants[a]

n	α_n^2	A_n $\beta = -1$	A_n $\beta = 1$	$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$ [b]
1	1.46718338	1.0814397	1.0133453	−1.22141980	0.19089535
2	3.65631416	−1.0149850	−0.98565571	1.83821967	−0.35838319(1)
3	5.85299460	0.84716695	0.83434604	−2.31280076	0.10093068(1)
4	8.05035049	−0.73430580	−0.72796463	2.70872568	−0.36785875(2)
5	10.24820111	0.65474875	0.65113018	−3.05469574	0.16575740(2)
6	12.44638556	−0.59558456	−0.59332308	3.36564645	−0.85483512(3)
7	14.64479511	0.54964259	0.54812447	−3.65036951	0.48808036(3)
8	16.84336139	−0.51273927	−0.51166619	3.91452249	−0.30002866(3)
9	19.04204128	0.48230032	0.48150963	−4.16198968	0.19553475(3)
10	21.24080718	−0.45665705	−0.45605574	4.39557035	−0.13334189(3)
11	23.43964125	0.43468237	0.43421219	−4.61735860	0.94415849(4)
12	25.63853211	−0.41558261	−0.41520722	4.82896783	−0.68897879(4)
13	27.83747279	0.39878739	0.39848192	−5.03167079	0.51623741(4)
14	30.03645955	−0.38386708	−0.38361482	5.22649088	−0.39496246(4)
15	32.23549108	0.37050295	0.37029155	−5.41426380	0.30836503(4)
16	34.43456804	−0.35843776	−0.35825884	5.59568046	−0.24421464(4)
17	36.63369271	0.34748174	0.34732836	−5.77131764	0.19678537(4)
18	38.83286884	−0.33746679	−0.33733457	5.94166023	−0.15992564(4)
19	41.03210147	0.32827672	0.32816127	−6.10711770	0.13219572(4)
20	43.23139686	−0.31979335	−0.31969246	6.26803641	−0.10953274(4)

[a] $Pe = 1, s = 0.25, \beta = -1, 1, \delta = 1.$
[b] $a(b)$ is equivalent to $a \times 10^{-b}$.

TABLE II
Eigenvalues and Related Constants[a]

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1-\eta^{s+1/s})\eta R_n(\eta)\,d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	4.41835430	1.0836486	1.0150909	−1.16928925	0.18848187
2	14.69645771	−1.0465902	−1.0142611	1.71764143	−0.38967573(1)
3	25.52923323	0.90766785	0.89139469	−2.18581790	0.12718920(1)
4	36.44125242	−0.79650587	−0.78775327	2.58874398	−0.50496964(2)
5	47.38265662	0.70954475	0.70443189	−2.94335360	0.23323678(2)
6	58.33986321	−0.64201864	−0.63883165	3.26213960	−0.12009012(2)
7	69.30684856	0.58878348	0.58667720	−3.55356690	0.67559065(3)
8	80.28034047	−0.54594419	−0.54448423	3.82341292	−0.40747577(3)
9	91.25838188	0.51075254	0.50969867	−4.07514485	0.26026242(3)
10	102.23973098	−0.48129834	−0.48051204	4.31352414	−0.17410355(3)
11	113.22356562	0.45624264	0.45563915	−4.53897406	0.12108788(3)
12	124.20932365	−0.43462429	−0.43415027	4.75380820	−0.86943438(4)
13	135.19661121	0.41574781	0.41536777	−4.95937680	0.64191782(4)
14	146.18514764	−0.39980770	−0.39877799	5.15676364	−0.48474034(4)
15	157.17473105	0.38425548	0.38399903	−5.34685199	0.37394135(4)
16	168.16521606	−0.37093827	−0.37072352	5.53037069	−0.29306026(4)
17	179.15649914	0.35890690	0.35872459	−5.70792708	0.23381324(4)
18	190.14850859	−0.34795901	−0.34780323	5.88003096	−0.18843013(4)
19	201.14119768	0.33795603	0.33782109	−6.04711232	0.15445298(4)
20	212.13453977	−0.32875683	−0.32863977	6.20953463	−0.12712568(4)

[a] $Pe = 5, s = 0.25, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE III
Eigenvalues and Related Constants[a]

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1-\eta^{s+1/s})\eta R_n(\eta)\,d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	5.45662793	1.0818683	1.0132102	−1.15163617	0.18765648
2	22.75181060	−1.0657784	−1.0314833	1.63098317	−0.40910265(1)
3	43.24213628	0.95669023	0.93742256	−2.06619596	0.14959243(1)
4	64.49656221	−0.85650831	−0.84522015	2.46133575	−0.64839537(2)
5	86.03556987	0.76856597	0.76165736	−2.81802430	0.31420052(2)
6	107.71643154	−0.69553219	−0.69113317	3.14196991	−0.16539150(2)
7	129.48047474	0.63579221	0.63287343	−3.43915082	0.93462052(3)
8	151.29809615	−0.58684305	−0.58483155	3.71451072	−0.56067903(3)
9	173.15240021	0.54634470	0.54490903	−3.97187818	0.35418948(3)
10	195.03295531	−0.51241873	−0.51136216	4.21418403	−0.23375154(3)
11	216.93295744	0.48363089	0.48283179	−4.44368924	0.16023341(3)
12	238.84778198	−0.45889839	−0.45827976	4.66216280	−0.11341029(3)
13	260.77418235	0.43741230	0.43692323	−4.87101027	0.82573624(4)
14	282.70981963	−0.41855121	−0.41815781	5.07136499	−0.61552087(4)
15	304.65297338	0.40184940	0.40152765	−5.26415273	0.46901973(4)
16	326.60235725	−0.38693204	−0.38666561	5.45013829	−0.36352826(4)
17	348.55699755	0.37352147	0.37329769	−5.62995920	0.28696067(4)
18	370.51615118	−0.36137623	−0.36118677	5.80415069	−0.22913917(4)
19	392.47924887	0.35032739	0.35016478	−5.97316420	0.18608429(4)
20	414.44585505	−0.34020864	−0.34006865	6.13738140	−0.15201173(4)

[a] $Pe = 10, s = 0.25, \beta = -1, \delta = 1$.
[b] See footnote b to Table I.

TABLE IV
Eigenvalues and Related Constants[a]

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	1.72308842	1.1141621	0.98575368	−1.19937227	0.16911895
2	4.33833565	−1.0297050	−0.99271760	1.82953350	−0.21828935(1)
3	6.95235941	0.85586858	0.84436928	−2.30812473	0.43975494(2)
4	9.56763163	−0.73928548	−0.73369243	2.70545441	−0.15588930(2)
5	12.18377719	0.65774405	0.65482598	−3.05208399	0.64105511(3)
6	14.80045769	−0.59761943	−0.59578107	3.36341300	−0.33203497(3)
7	17.41747801	0.55110439	0.54991914	−3.64838350	0.18201343(3)
8	20.03472594	−0.51386435	−0.51301725	3.91271325	−0.11296181(3)
9	22.65213432	0.48319259	0.48258295	−4.16031469	0.71905015(4)
10	25.26966152	−0.45739319	−0.45692477	4.39400187	−0.49483457(4)
11	27.88728111	0.43530018	0.43493981	−4.61587741	0.34492865(4)
12	30.50497630	−0.41611394	−0.41582388	4.82755998	−0.25363153(4)
13	33.12273666	0.39924939	0.39901599	−5.03032577	0.18794779(4)
14	35.74055628	−0.38427533	−0.38408138	5.22520051	−0.14464138(4)
15	38.35843261	0.37086660	0.37070534	−5.41302158	0.11206698(4)
16	40.97636568	−0.35876528	−0.35862823	5.59448111	−0.89085108(5)
17	43.59435770	0.34777872	0.34766179	−5.77015680	0.71472216(5)
18	46.21241271	−0.33773809	−0.33763714	5.94053423	−0.58133020(5)
19	48.83053640	0.32852568	0.32843763	−6.10602343	0.48036004(5)
20	51.44873599	−0.32002165	−0.31994489	6.26697122	−0.39665366(5)

[a] $Pe = 1, s = 0.5, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE V
Eigenvalues and Related Constants[a]

n	α_n^2	A_n $\beta = -1$	A_n $\beta = 1$	$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta) d\eta$[b]
1	4.96970125	1.1225862	0.99190190	−1.11306291	0.16497267
2	17.22544865	−1.0993984	−1.0507750	1.67770960	−0.28194329(1)
3	30.07976324	0.95383670	0.93408025	−2.16126445	0.74472866(2)
4	43.03969974	−0.82488964	−0.81533010	2.57178363	−0.26454648(2)
5	56.04548710	0.72725341	0.72227508	−2.93017327	0.10875558(2)
6	69.07669538	0.65353974	−0.65059201	3.25106894	−0.53073420(3)
7	82.12347282	0.59676840	0.59492883	−3.54381835	0.28176771(3)
8	95.18043572	−0 55177182	−0.55052194	3.81457439	−0.16641204(3)
9	108.24441034	0.51521082	0.51433640	−4.06757861	0.10299649(3)
10	121.31341611	−0.48483491	−0.48418711	4.30588123	−0.68374370(3)
11	134.38616026	0.45913452	0.45864683	−4.53175465	0.46643396(4)
12	147.46176829	−0.43704579	−0.43666352	4.74694167	−0.33411181(4)
13	160.53963194	0.41781567	0.41751313	−4.95281097	0.24350347(4)
14	173.61931925	−0.40088178	−0.40063530	5.15045861	−0.18378326(4)
15	186.70051927	0.38583288	0.38563060	−5.34077634	0.14052360(4)
16	199.78300672	−0.37234051	−0.37217118	5.52449911	−0.11007701(4)
17	212.86661894	0.36016542	0.36002245	−5.70223875	0.87347881(5)
18	225.95124028	−0.34909751	−0.34897555	5.87450851	−0.70264860(5)
19	239.03679141	0.33899304	0.33888758	−6.04174104	0.57500645(5)
20	252.12322183	−0.32970567	−0.32961460	6.20430190	−0.47101554(5)

[a] $Pe = 5, s = 0.50, \beta = -1, 1, \delta = 1.$
[b] See footnote b to Table 1.

TABLE VI
Eigenvalues and Related Constants[a]

n	ρ_n^2	A_n $\beta = -1$	A_n $\beta = 1$	$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\, \eta R_n(\eta) d\eta$[b]
1	5.99596553	1.1207104	0.98946636	−1.08735677	0.16371653
2	26.33911389	−1.1383503	−1.0825649	1.56705291	−0.31912989(1)
3	50.51754654	1.0333299	1.0059303	−2.01865948	0.10241581(1)
4	75.66014985	−0.91139605	−0.89694716	2.42676675	−0.39825644(2)
5	101.18759197	0.80610541	0.79820697	−2.79130003	0.17207811(2)
6	126.91666176	−0.72100625	−0.71635642	3.11988115	−0.83591452(3)
7	152.76655715	0.65365401	0.65078211	−3.41994811	0.43933903(3)
8	178.69477706	−0.59980780	−0.59791398	3.69723810	−0.25195870(3)
9	204.67662819	0.55614133	0.55484550	−3.95598728	0.15253260(3)
10	230.69677864	−0.52007238	−0.51914059	4.19934035	−0.98310934(4)
11	256.74522394	0.48979989	0.48911303	−4.42967595	0.65669242(4)
12	282.81517380	−0.46399462	−0.46346941	4.64883141	−0.45894714(4)
13	308.90186968	0.44171519	0.44130682	−4.85825418	0.32862180(4)
14	335.00188817	−0.42224715	−0.42192098	5.05910452	−0.24319166(4)
15	361.11271284	0.40507244	0.40480865	−5.25232633	0.18324389(4)
16	387.23246154	−0.38977673	−0.38955933	5.43869707	−0.14132893(4)
17	413.35970734	0.37605953	0.37587824	−5.61886342	0.11076267(4)
18	439.49335768	−0.36366038	−0.36350760	5.79336741	−0.88030613(5)
19	465.63257095	0.35239919	0.35226850	−5.96266569	0.71250549(5)
20	491.77669772	−0.34209860	−0.34198679	6.12714397	−0.57839651(5)

[a] $Pe = 10, s = 0.50, \beta = -1, 1, \delta = 1$.

[b] See footnote b to Table 1.

TABLE VII
Eigenvalues and Related Constants[a]

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	1.90652424	1.1349705	0.97003932	−1.18602560	0.15536593
2	4.82388988	−1.0368999	−1.0017173	1.82596199	−0.15201917(1)
3	7.73619337	0.85980055	0.84960016	−2.30601326	0.28378574(2)
4	10.65008955	−0.74147746	0.73652427	2.70383289	−0.99876888(3)
5	13.56507430	0.65910128	0.65663705	−3.05070964	0.39191826(3)
6	16.48071102	−0.59859096	−0.59699306	3.36219338	−0.20845902(3)
7	19.39675640	0.55182130	0.55083105	−3.64727288	0.10995855(3)
8	22.31307307	−0.51443951	−0.51370901	3.91168517	−0.70341893(4)
9	25.22957976	0.48365543	0.48314772	−4.15935218	0.43280568(4)
10	28.14622639	−0.45778617	−0.45738409	4.39309324	−0.30669182(4)
11	31.06298129	0.43563229	0.43533247	−4.61501416	0.20736648(4)
12	33.97982427	−0.41640538	−0.41615721	4.82673568	−0.15668343(4)
13	36.89674278	0.39950352	0.39930935	−5.02953540	0.11297000(4)
14	39.81372955	−0.38450318	−0.38433768	5.22444007	−0.89118375(5)
15	42.73078127	0.37106969	0.37093548	−5.41228780	0.67389184(5)
16	45.64789762	−0.35895018	−0.35883351	5.59377129	−0.54753873(5)
17	48.56508079	0.34794628	0.34784886	−5.76946865	0.43022286(5)
18	51.48233508	−0.33789242	−0.33780671	5.93986583	−0.35635562(5)
19	54.39966667	0.32866752	0.32859405	−6.10537311	0.28966141(5)
20	57.31708351	−0.32015277	−0.32008780	6.26633752	−0.24236140(5)

[a] $Pe = 1, s = 0.75, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE VIII
Eigenvalues and Related Constants[a]

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1-\eta^{s+1/s})\eta R_n(\eta)d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	5.37809946	1.1485119	0.97934575	−1.08144181	0.15046510
2	19.01446990	−1.1313696	−1.0765671	1.66036025	−0.23116686(1)
3	33.30755205	0.97524637	0.95462569	−2.15053231	0.56378509(2)
4	47.72885352	−0.83747003	−0.82809647	2.56383362	−0.18775014(2)
5	62.20863801	0.73495496	0.73032247	−2.92356607	0.73215796(3)
6	76.72087970	−0.65862688	−0.65592473	3.24524248	−0.35159100(3)
7	91.25287201	0.60039266	0.59875969	−3.53851510	0.18080124(3)
8	105.79770875	−0.55450580	−0.55339162	3.80965654	−0.10715380(3)
9	120.35134671	0.51736890	0.51660859	−4.06296275	0.64714057(4)
10	134.91127681	−0.48659689	−0.48602727	4.30151206	−0.43422723(4)
11	149.47586785	0.46061047	0.46019091	−4.52759298	0.28991818(4)
12	164.04401936	−0.43830845	−0.43797502	4.74295828	−0.21046865(4)
13	178.61496694	0.41891271	0.41865404	−4.94898329	0.15040934(4)
14	193.18816781	−0.40184931	−0.40163552	5.14676865	−0.11512643(4)
15	207.76323060	0.38669548	0.38652318	−5.33720945	0.86470318(5)
16	222.33987084	−0.37310850	−0.37296227	5.52104314	−0.68651833(5)
17	236.91788178	0.36085297	0.36073146	−5.69888346	0.53632298(5)
18	251.49711476	−0.34973388	−0.34962897	5.87124516	−0.43647891(5)
19	266.07746584	0.33957864	0.33948907	−6.03856208	0.35277302(5)
20	280.65886633	−0.33024394	−0.33016593	6.20120077	−0.29138271(5)

[a] $Pe = 5, s = 0.75, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE IX
Eigenvalues and Related Constants[a]

n	α_n^2	A_n $\beta = -1$	A_n $\beta = 1$	$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
1	6.41656735	1.1469328	0.97670655	−1.05227999	0.14906948
2	28.87636181	−1.1817042	−1.1152076	1.53886273	−0.27614031(1)
3	55.66428461	1.0697892	1.0387906	−1.99796344	0.84061791(2)
4	83.57009882	−0.93675313	−0.92135226	2.41116463	−0.30756319(2)
5	111.93850290	0.82287399	0.81489426	−2.77853602	0.12584874(2)
6	140.55391064	−0.73228806	−0.72775273	3.10875387	−0.58967898(3)
7	169.31829227	0.66163022	0.65892428	−3.40986551	0.29932318(3)
8	198.17940171	−0.60571197	−0.60395218	3.68789172	−0.16920607(3)
9	227.10675151	0.56072080	0.55954401	−3.94720092	0.10011557(3)
10	256.08139169	−0.52374601	−0.52290191	4.19100372	−0.64343774(4)
11	285.09095600	0.49283912	0.49222782	−4.42171437	0.42231170(4)
12	314.12705223	−0.46656344	−0.46609456	4.64119108	−0.29597962(4)
13	343.18379939	0.44392945	0.44357039	−4.85089439	0.20876320(4)
14	372.25696571	−0.42418253	−0.42389408	5.05199330	−0.15534966(4)
15	401.34343897	0.40678625	0.40655611	−5.24543786	0.11549426(4)
16	430.44088931	−0.39130890	−0.39111796	5.43200997	−0.89657064(5)
17	459.54754787	0.37744197	0.37728463	−5.61235970	0.69447209(5)
18	488.66205741	−0.36491637	−0.36478294	5.78703168	−0.55529756(5)
19	517.78336932	0.35354827	0.35343522	−5.95648464	0.44527330(5)
20	546.91067091	−0.34315449	−0.34305735	6.12110596	−0.36294222(5)

[a] $Pe = 10, s = 0.75, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE X
Eigenvalues and Related Constants

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	2.23764093	1.1646754	0.95000968	−1.16670551	0 13475595
2	5.69541644	−1.0454751	−1.0203520	1.82036520	−0.76296209(2)
3	9.14489234	0.86556796	0.85540216	−2.30267966	0.18915362(2)
4	12.59647467	−0.74404745	−0.74098136	2.70129330	−0.42591216(3)
5	16.04945041	0.66113855	0.65877021	−3.04857701	0.25262596(3)
6	19.50325077	−0.59981765	−0.59885367	3.36031409	−0.86132458(4)
7	22.95756522	0.55291634	0.55197777	−3.64557040	0.70061460(4)
8	26.41221990	−0.51520211	−0.51476388	3.91011532	−0.28816113(4)
9	29.86711236	0.48436555	0.48388872	−4.15788676	0.27366786(4)
10	33.32217955	−0.45832298	−0.45832298	4.39171296	−0.12537737(4)
11	36.77738155	0.43614169	0.43586205	−4.61370512	0.13034869(4)
12	40.23269286	−0.41681114	−0.41666211	4.82548744	−0.64054883(5)
13	43.68809748	0.39989259	0.39971246	−5.02833990	0.70680978(5)
14	47.14358601	−0.38482451	−0.38472499	5.22329090	−0.36446020(5)
15	50.59915390	0.37137990	0.37125590	−5.41117974	0.42013590(5)
16	54.05480036	−0.35921322	−0.35914304	5.59270006	−0.22382300(5)
17	57.51052762	0.34820158	0.34811183	−5.76843066	0.26760130(5)
18	60.96634051	−0.33811329	−0.33806180	5.93885803	−0.14533542(5)
19	64.42224616	0.32888316	0.32881555	−6.10439289	0.18002604(5)
20	67.87825376	−0.32034176	−0.32030290	6.26538263	−0.98305593(6)

[a] $Pe = 1, s = 1.5, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE XI
Eigenvalues and Related Constants[a]

n	α_n^2	A_n, $\beta = -1$	A_n, $\beta = 1$	$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
1	6.13131731	1.1871769	0.96423477	−1.03827912	0.12918272
2	22.020845924	−1.1774300	−1.1180980	1.63633022	−0.17086142(1)
3	39.08402911	1.0018577	0.98033930	−2.13535996	0.39765931(2)
4	56.13381246	−0.85361045	−0.84550425	2.55225868	−0.11043433(2)
5	73.26565355	0.74458528	0.74026467	−2.91378030	0.46092674(3)
6	90.44282511	−0.66526320	−0.66317382	3.23654291	−0.18473958(3)
7	107.64739932	0.60517427	0.60369740	−3.53056783	0.11034886(3)
8	124.86971081	−0.55821157	−0.55739646	3.80227508	−0.53253271(4)
9	142.10413737	0.52034044	0.51965842	−4.05603014	0.39181737(4)
10	159.34720490	−0.48905273	−0.48864945	4.29494867	−0.20878775(4)
11	176.59666072	0.46269724	0.46232181	−4.52134132	0.17515165(4)
12	193.85098689	−0.44009931	−0.43986839	4.73697502	−0.98971370(5)
13	211.10912908	0.42048768	0.42025639	−4.94323461	0.90823213(5)
14	228.37033766	−0.40323379	−0.40308811	5.14122751	−0.53255238(5)
15	245.63407058	0.38793943	0.38778537	−5.33185373	0.52225384(5)
16	262.89993181	−0.37422953	−0.37413119	5.51585443	−0.31336750(5)
17	280.16763117	0.36188546	0.36177673	−5.69384625	0.32420392(5)
18	297.43695728	−0.35066130	−0.35059159	5.86634616	−0.19682895(5)
19	314.70775905	0.34043513	0.34035486	−6.03378984	0.21363682(5)
20	331.97993268	−0.33102956	−0.33097836	6.19654530	−0.12973722(5)

[a] $Pe = 5, s = 1.5, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE XII
Eigenvalues and Related Constants[a]

n	α_n^2	A_n, $\beta = -1$	A_n, $\beta = 1$	$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
1	7.21452427	2.3715027	0.96143109	−1.00542351	0.12771858
2	33.40512380	−1.7254973	−1.1642255	1.50145760	−0.22272741(1)
3	64.85566400	1.3230666	1.0804948	−1.97045382	0.64445763(2)
4	97.71695737	−1.0683054	−0.95389553	2.38982909	−0.21146925(2)
5	131.18736010	0.89141586	0.83572908	−2.76054804	0.84212872(3)
6	164.98937605	−0.77332297	−0.74267025	3.09273607	−0.35652021(3)
7	198.99270601	0.68699602	0.66928597	−3.39515959	0.18650419(3)
8	233.12705996	−0.62339938	−0.61204328	3.67415061	−0.94976781(4)
9	267.35125011	0.57323684	0.56574694	−3.93421927	0.60580944(4)
10	301.63963162	−0.53341435	−0.52808797	4.17864739	−0.34524787(4)
11	335.97545157	0.50031441	0.49647137	−4.40988883	0.25256058(4)
12	370.34732974	−0.47271533	−0.46979333	4.62982582	−0.15397799(4)
13	404.74728272	0.44896859	0.44672445	−4.83993460	0.12426113(4)
14	439.16595586	−0.42849885	−0.42671575	5.04139500	−0.78950544(5)
15	473.60992091	0.41046367	0.40902937	−5.23516487	0.68636582(5)
16	508.06519178	−0.39454285	−0.39336961	5.42203200	−0.44703497(5)
17	542.53925271	0.38027662	0.37929713	−5.60265094	0.41279868(5)
18	577.01134209	−0.36745403	−0.36663834	5.77756991	−0.27225353(5)
19	611.49891550	0.35582202	0.35511816	−5.94725047	0.26511159(5)
20	645.99454712	−0.34521421	−0.34462332	6.11208235	−0.17507644(5)

[a] $Pe = 10, s = 1.5, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE XIII
Eigenvalues and Related Constants[a]

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	2.36663878	2.3093604	0.94461674	−1.16052888	0.12800064
2	6.03375460	−1.2535887	−1.0275681	1.81823891	−0.56645982(2)
3	9.69213769	0.92549061	0.85679867	−2.30137770	0.17885992(2)
4	13.35284356	−0.77109272	−0.74267004	2.70031614	−0.23635508(8)
5	17.01505306	0.67541706	0.65929548	−3.04777271	0.24757685(3)
6	20.67814855	−0.60856851	−0.59956400	3.35961791	−0.48285949(4)
7	24.34179573	0.55871259	0.55227584	−3.64494892	0.70244347(4)
8	28.00580792	−0.51925770	−0.51516413	3.90954900	−0.15161427(4)
9	31.67007522	0.48738035	0.48408944	−4.15736317	0.27863545(4)
10	35.33452996	−0.46059149	−0.45834397	4.39122365	−0.63084077(5)
11	38.99912930	0.43795010	0.43600987	−4.61324407	0.13421292(4)
12	42.66384584	−0.41823502	−0.41684990	4.82505019	−0.31152141(5)
13	46.32866235	0.40108279	0.39982752	−5.02792304	0.73408682(5)
14	49.99356867	−0.38579029	−0.38486772	5.22289174	−0.17234607(5)
15	53.65855984	0.37221622	0.37134895	−5.41079616	0.43937417(5)
16	57.32363490	−0.35990559	−0.35925617	5.59233030	−0.10316350(5)
17	60.98879612	0.34881881	0.34818923	−5.76807327	0.28143249(5)
18	64.65404857	−0.33863045	−0.33815433	5.93851180	−0.65232579(6)
19	68.31939973	0.32935675	0.32888146	−6.10405678	0.19020078(5)
20	71.98485932	−0.32074027	−0.32038051	6.26505575	−0.42758159(6)

[a] $Pe = 1, s = 2, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE XIV
Eigenvalues and Related Constants[a]

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	6.42841891	2.3811528	0.96049011	−1.02505678	0.12231769
2	23.44464520	−1.5925538	−1.1323915	1.62832841	−0.15428490(1)
3	41.32199599	1.1389102	0.98799249	−2.13004338	0.35979750(2)
4	59.39308283	−0.91528980	−0.85120551	2.54811134	−0.90722386(3)
5	77.55577903	0.77316082	0.74317434	−2.91026310	0.41418788(3)
6	95.76895389	−0.68258041	−0.66558658	3.23342915	−0.14116156(3)
7	114.01258524	0.61539717	0.60517998	−3.52774087	0.10102328(3)
8	132.27590881	−0.56553625	−0.55875144	3.79966580	−0.38359971(4)
9	150.52267413	0.52531729	0.52059609	−4.05359356	0.36621689(4)
10	168.83902284	−0.49295521	−0.48954324	4.29265353	−0.14299438(4)
11	187.13245427	0.46559184	0.46298780	−4.51916484	0.16670109(4)
12	205.43128363	−0.44249168	−0.44051545	4.73489995	−0.64874889(5)
13	223.73434047	0.42237115	0.42076304	−4.94124753	0.87768531(5)
14	242.04079211	−0.40484240	−0.40358516	5.13931769	−0.33580289(5)
15	260.35003590	0.38926234	0.38818860	−5.33001248	0.51119827(5)
16	278.66163085	−0.37538336	−0.37452897	5.51407454	−0.19073979(5)
17	296.97525304	0.36286753	0.36210815	−5.69212167	0.32081023(5)
18	315.29066566	−0.35152892	−0.35091966	5.86467177	−0.1158390(5)
19	333.60769844	0.34119565	0.34063403	−6.03216122	0.21337696(5)
20	351.92623325	−0.33170555	−0.33125532	6.19495864	−0.73775786(6)

[a] $Pe = 5, s = 2, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table 1.

TABLE XV
Eigenvalues and Related Constants[2]

n	α_n^2	A_n		$dR_n(1)/d\eta$	$\int_0^1 (1 - \eta^{s+1/s})\eta R_n(\eta)\, d\eta$[b]
		$\beta = -1$	$\beta = 1$		
1	7.53394877	2.3916131	0.95767524	−0.99128103	0.12086032
2	35.15844867	−1.7721108	−1.1803893	1.48961075	−0.20726655(1)
3	68.41254179	1.3417580	1.0930888	−1.96147104	0.59296723(2)
4	103.19551949	−1.0814303	−0.96418823	2.38265101	−0.18655298(2)
5	138.64642148	0.89743522	0.84192166	−2.75438322	0.75045859(3)
6	174.46273036	−0.77817981	−0.74739714	3.08720242	−0.29890439(3)
7	210.50124709	0.68981431	0.67240930	−3.39006853	0.16513492(3)
8	246.68450481	−0.62596686	−0.61465856	3.66939678	−0.76460021(4)
9	282.96706125	0.57499015	0.56765986	−3.92973640	0.54073442(4)
10	319.32060513	−0.53507409	−0.52979109	4.17438999	−0.26820232(4)
11	355.72663217	0.50156456	0.49780351	−4.40582363	0.22832199(4)
12	392.17256619	−0.47390985	−0.47101940	4.62592739	−0.11573471(4)
13	428.64958025	0.44992646	0.44772675	−4.83618286	0.11386494(4)
14	465.15131122	−0.42941561	−0.42765586	5.03777367	−0.57528922(5)
15	501.67306933	0.41123094	0.40982196	−5.23166052	0.63715076(5)
16	538.21133451	−0.39527718	−0.39412187	5.41863334	−0.31625612(5)
17	574.76342538	0.38091046	0.37994595	−5.59934837	0.38783353(5)
18	611.32727585	−0.36806059	−0.36725911	5.77435516	−0.18714392(5)
19	647.90128073	0.35635794	0.35566313	−5.94411634	0.25183840(5)
20	684.48418676	−0.34572706	−0.34514775	6.10902252	−0.11689646(5)

[a] $Pe = 10\ s = 2, \beta = -1, 1, \delta = 1$.
[b] See footnote b to Table I.

TABLE XVI
Comparison of Eigenvalues Obtained from Present Analysis with Previous Workers for Extreme Case $Pe \to \infty, \beta = \delta = 0$

	$s = 1$				$s = 0.5$		$s = 1/30$	
n	Present analysis	Lyche and Bird[8]	Sellars, Tribus, and Klein[11]	Abramowitz[1]	Present analysis	Lyche and Bird[8]	Present analysis	Lyche and Bird[8]
1	7.31358692	7.314	7.1129	7.3135868	6.58236351	6.582	6.26298320	6.263
2	44.60946123	44.61	44.89	44.609468	39.09337938	39.09	36.35960340	36.35
3	113.92103315	113.9	113.785	113.92104	99.49622302	99.50	92.32598775	92.34
4	215.24056099	215.25	215.121	215.24059	187.79530099	187.9	174.14059533	
5	348.56419555		348.457	348.56405	303.98658840		281.79849970	
6	513.89032840		513.793		448.06836320		415.29815562	
7	711.21826434		711.129		620.03987657		574.63893447	
8	940.54778743		940.465		819.90085695		759.82061774	
9	1201.87901760		1201.8		1047.65133960		970.84324599	
10	1495.21237594		1495.1		1303.29161571		1207.70707660	
11	1820.54859923				1586.82223228		1470.41258522	
12	2177.88877962				1898.24401745		1758.96048833	
13	2567.23442083				2237.55812069		2073.35177852	
14	2988.58750534				2604.76606301		2413.58776795	
15	3441.95057072				2999.86979479		2779.67013854	
16	3927.32679341				3422.87175985		3171.60099710	
17	4444.72007925				3873.77496496		3589.38293562	
18	4994.13516006				4352.58305429		4033.01909552	
19	5575.57769586				4859.30038840		4502.51323590	
20	6189.05438227				5393.93212738		4997.86980551	

TABLE XVII

Comparison of Eigenfunctions Obtained from Present Analysis with Those of Lyche and Bird[8] for Extreme Cases $Pe \to \infty, \beta = \delta = 0$

	$s = 1$						$s = 0.5$						$s = 1/3$					
	Present analysis			Lyche and Bird			Present Analysis			Lyche and Bird			Present Analysis			Lyche and Bird		
η	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3
0	1.000000	1.000000	1.000000	1.000	1.000	1.000	1.000000	1.000000	1.000000	1.000	1.000	1.000	1.000000	1.000000	1.000000	1.000	1.000	1.000
0.1	0.981845	0.891809	0.735450	0.982	0.892	0.735	0.983614	0.904643	0.766341	0.988	0.884	0.766	0.984404	0.911147	0.782169	0.984	0.911	0.782
0.2	0.928893	0.604700	0.152473	0.929	0.605	0.153	0.935333	0.646046	0.227448	0.935	0.646	0.288	0.938355	0.668200	0.269282	0.938	0.668	0.269
0.3	0.845468	0.233857	−0.315213	0.846	0.234	−0.315	0.857876	0.298037	−0.256352	0.855	0.310	−0.257	0.864086	0.331584	−0.217505	0.854	0.335	−0.218
0.4	0.738094	−0.109593	−0.392084	0.738	−0.110	−0.392	0.755835	−0.044010	−0.403870	0.756	−0.045	−0.403	0.765347	−0.002432	−0.404454	0.765	−0.002	−0.404
0.5	0.614599	−0.342141	−0.142342	0.615	−0.342	−0.142	0.635411	−0.296272	−0.208450	0.638	−0.311	−0.209	0.647344	−0.263997	−0.248937	0.647	−0.264	−0.249
0.6	0.483097	−0.432182	0.169685	0.483	−0.432	0.170	0.503861	−0.413795	0.107543	0.506	−0.414	0.118	0.516511	−0.398810	0.062004	0.517	−0.399	0.062
0.7	0.351010	−0.397629	0.331488	0.351	−0.398	0.332	0.368680	−0.399769	0.306858	0.374	−0.385	0.310	0.379998	−0.400026	0.285506	0.380	−0.400	0.286
0.8	0.224264	−0.284494	0.302723	0.224	−0.285	0.303	0.236647	−0.293825	0.305486	0.236	−0.293	0.310	0.244868	−0.300543	0.306354	0.245	−0.301	0.306
0.9	0.106741	−0.141133	0.162625	0.107	−0.141	0.163	0.112876	−0.147478	0.169261	0.110	−0.166	0.185	0.177025	−0.152412	0.174476	0.117	−0.153	0.174
1.0	0.000000	0.000000	0.000000	0.000	0.000	0.000	0.000000	0.000000	0.000000	0.000	0.000	0.000	0.000000	0.000000	0.000000	0.000	0.000	0.000

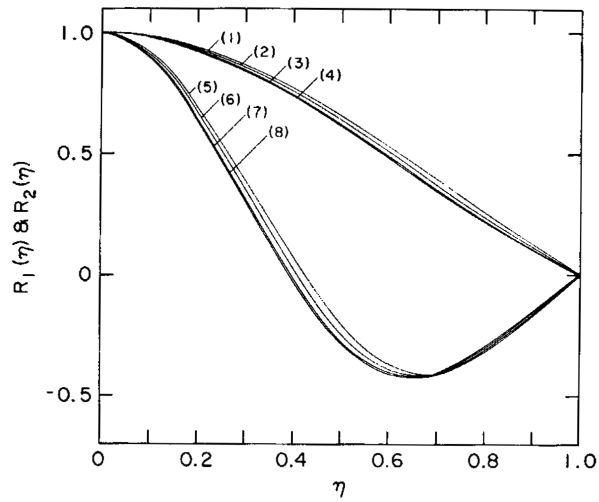

Fig. 1. First two eigenfunctions in non-Newtonian tube flow (Pe = 5) for different values of the power law index: $R_1(\eta)$: (1) $s = 0.25$, (2) $s = 0.75$, (3) $s = 1.5$; (4) $s = 2$; $R_2(\eta)$: (5) $s = 0.25$; (6) $s = 0.75$; (7) $s = 1.5$; (8) $s = 2$.

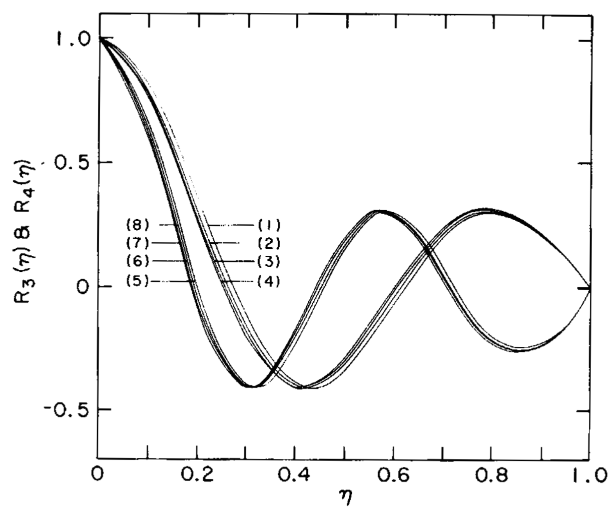

Fig. 2. Third and fourth eigenfunctions in non-Newtonian tube flow (Pe = 5) for different values of the power law index: $R_3(\eta)$: (1) $s = 0.25$; (2) $s = 0.75$; (3) $s = 1.5$; (4) $s = 2$; $R_4(\eta)$: (5) $s = 0.25$; (6) $s = 0.75$; (7) $s = 1.5$; (8) $s = 2$.

In addition, if one defines the Nusselt number as

$$Nu = \frac{-2a \left.\dfrac{\partial T}{\partial r}\right|_{r=a}}{T_{av} - T_w}$$

one obtains

$$Nu = \frac{2}{\theta_b(\xi)}\left[\frac{\beta s}{2s+1} + \delta - \sum_{n=1}^{\infty} A_n \frac{dR_n(1)}{d\eta} \exp(-\alpha_n^2 \xi)\right] \tag{21}$$

Before proceeding further, it is desirable to calculate the simplified results from the present analysis and compare them with those of previous workers. The dimensionless average temperature versus axial distance along the tube for $Pe \to \infty$ and $\beta = \delta = 0$ is shown in Figure 3 in a comparison of the present result with those of Lyche and Bird.[8] Exact agreement is obtained for the two methods. [It should be noted that Lyche and Bird obtained the eigenfunctions $R_n(\eta)$ by expanding them in terms of a series expansion with respect to the dimensionless radial coordinate η, a method different from the one used here.] Comparison has also been made of radial temperature distributions obtained by the two methods at various axial positions and good agreement is shown.

Figure 4 shows the effect of β, which is directly related to the Brinkman number, on the dimensionless average temperature distribution and on the Nusselt number. As β increases from -1 to 1 at 0.5 intervals, the dimensionless average temperature also increases. When the value of β increases, heat generation in the fluid due to viscous dissipation also increases, and therefore the temperature of the fluid is expected to rise. Since heat is constantly transferred to the wall from the fluid, the temperature of the fluid decreases along the length of the tube. From the curves of the Nusselt number to the axial distance of the tube, one can see that a high heat transfer rate occurs in the entrance region of the tube and decreases to a constant value further downstream. The Nusselt number is also larger for a larger value of the parameter β. A higher β value corresponds to a larger temperature difference between the fluid and the wall and consequently a higher heat transfer rate.

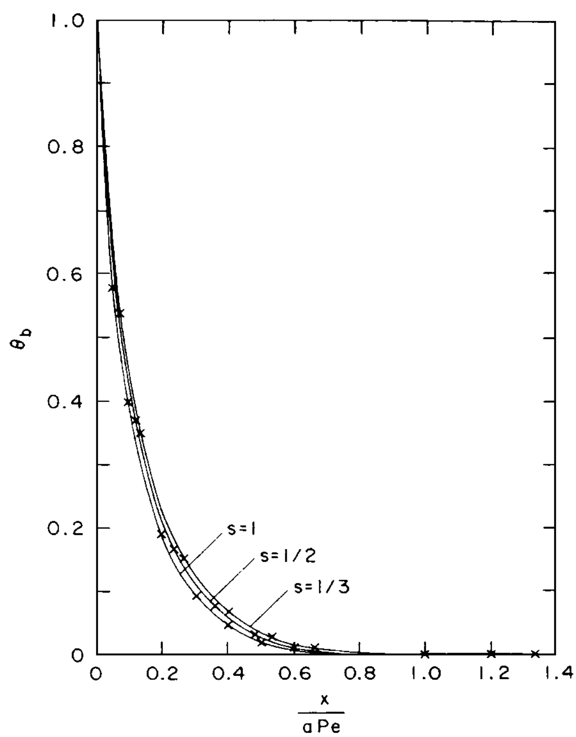

Fig. 3. Comparison of the present simplified results with those of Lyche and Bird[8] for the dimensionless average temperature vs. axial distance for different values of the power law index: $Pe = 10^8$; $\beta = \delta = 0$; (—) present results; (×) Lyche and Bird.

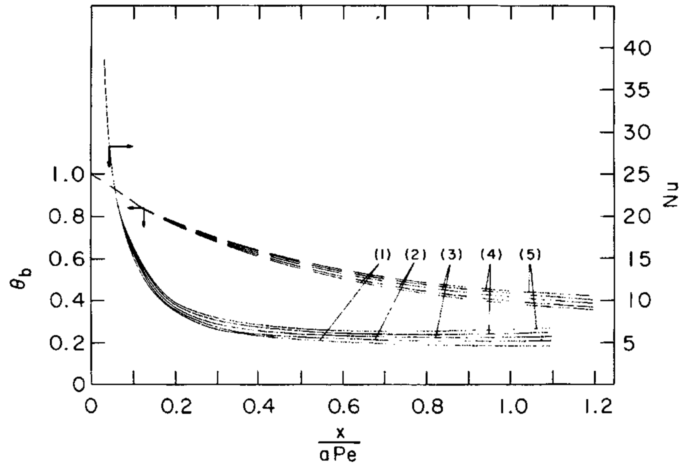

Fig. 4. Dimensionless average temperature and Nusselt number vs axial distance for various Brinkman numbers: $Pe = 1$; $\delta = 1$; $s = 0.75$; (1) $\beta = -1.0$; (2) $\beta = -0.5$; (3) $\beta = 0$; (4) $\beta = 0.5$; (5) $\beta = 1.0$.

Figure 5 shows the effects of β and δ on the dimensionless average temperature and the Nusselt number. Some effects are obtained relating to the range of parameters investigated, $\beta = -1, -0.5, 0, 0.5, 1$ and $\delta = -1$ and 1. A positive value of δ is equivalent to a heat source, while a negative value of δ is equivalent to a heat sink in the system. When $\delta = -1$ and for all the values of β examined, the dimensionless average temperature θ_b can decrease to a negative value further downstream in the tube. Since a negative value of δ is equivalent to a heat sink in the fluid, the temperature of the fluid may decrease to a value lower than that of the temperature of the wall. This then causes θ_b to be negative. The Nusselt number also decreases along the length of the tube and may become negative at the location where $\theta_b \to 0$. In the region where $\theta_b < 0$, both the temperature difference between the fluid and the wall and the temperature gradient at the

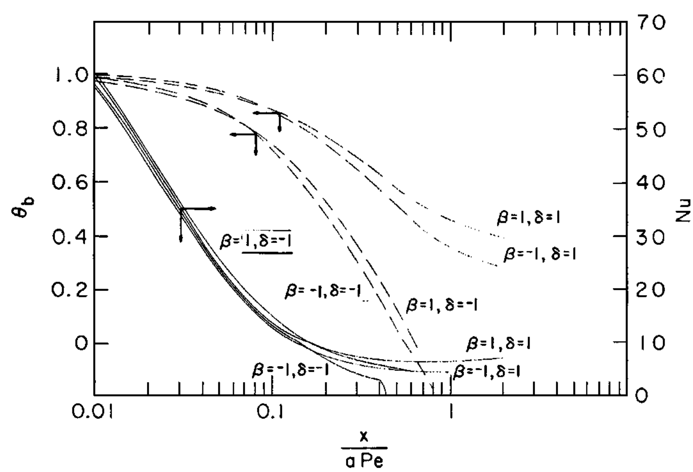

Fig. 5. Dimensionless average temperature and Nusselt number vs axial distance for various values of β and δ: $Pe = 1$; $\delta = 1.5$.

wall change sign so that the Nusselt number becomes negative. This phenomenon was also observed by Vlachopoulos and Keung.[13] When $\delta = 1$, the average dimensionless temperature of the fluid decreases to a positive constant value, and the corresponding Nusselt number also approaches the constant value. When both β and δ are negative, they act like a sink in the fluid so that the temperature of the fluid can decrease to a value lower than that of the wall temperature.

Figures 6 to 8 show the dimensionless average temperature and the Nusselt number for $Pe = 1, 5$, and 10, and for $s = 0.25, 0.5, 0.75$, and 2. For any Peclet number investigated, the dimensionless average temperature and the Nusselt number increases as the power law index s decreases ($s < 1$ corresponds to a pseudoplastic fluid, $s > 1$ corresponds to a dilatant fluid). This implies that a pseudoplastic fluid transfers more heat than a dilatant fluid. When s is kept constant, the dimensionless average temperature downstream in the tube in-

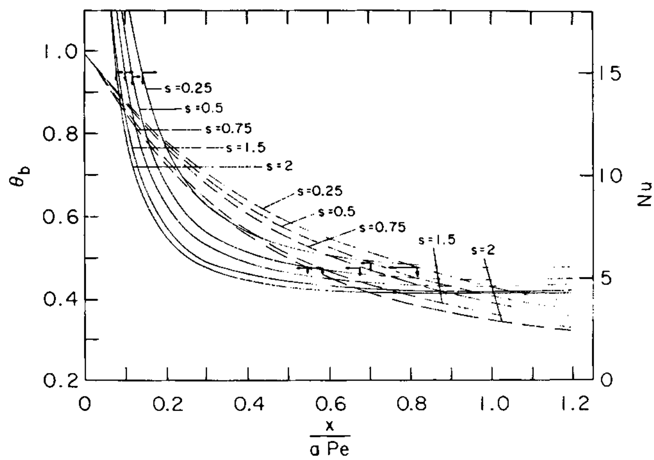

Fig. 6. Dimensionless average temperature and Nusselt number vs axial distance for various values of the power law index for flow in a tube (Pe - 1, $\delta = 1, \beta = -1$): $Pe = 1; \delta = 1; \beta = -1$.

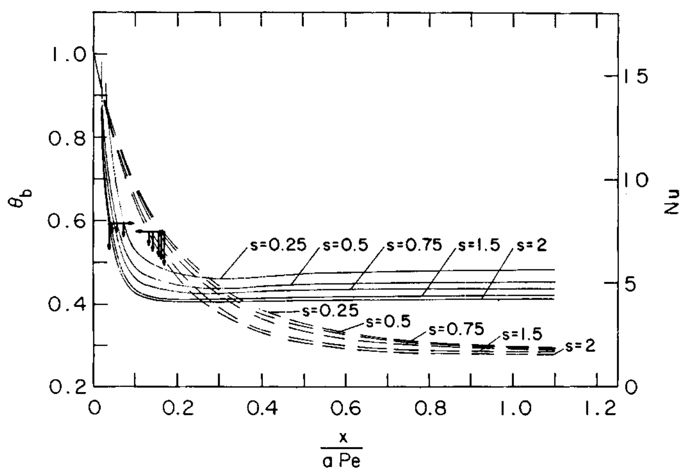

Fig. 7. Dimensionless average temperature and Nusselt number vs axial distance for various values of the power law index for flow in a tube ($Pe = 5, \delta = 1, \beta = -1$): $Pe = 5; \delta = 1; \beta = -1$.

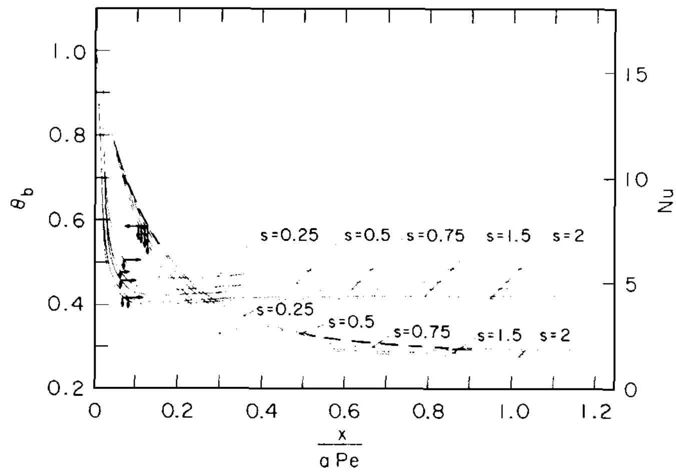

Fig. 8. Dimensionless average temperature and Nusselt number vs axial distance for various values of the power law index for flow in a tube ($Pe = 10, \delta = 1, \beta = -1$): $Pe = 10$; $\delta = 1$; $\beta = -1$.

creases as the Peclet number decreases because of axial conduction. For Pe = 1 to 10, the Nusselt number remains at about the same value for $\xi \gtrsim 1$. However, at a small value of ξ, the Nusselt number becomes larger as the Pe decreases, which indicates a strong effect of axial conduction on the Nusselt number at the entrance region.

SUMMARY AND CONCLUSION

From the present study it is found that (1) the heat transfer rate increases when the Brinkman number increases; (2) when values of β and δ are negative, the system acts like a sink and the dimensionless average temperature can be lower than the dimensionless wall temperature; (3) the heat transfer rate increases as the fluid changes from dilatant ($s > 1$) to pseudoplastic ($s < 1$); and (4) when axial conduction increases (Peclet number decreases), a higher dimensionless average temperature is obtained.

Nomenclature

a	radius of tube
A_n	coefficient of series expansion in eq. (11)
C_p	heat capacity
h	heat transfer coefficieent
I_n	integral defined by eq. (17)
J_n	integral defined by eq. (18)
k	thermal conductivity
m	constant in power law model
Nu	Nusselt number, eq. (21)
Pe	$aU_{av}\rho C_p/k$
q	heat generation term
r	radial coordinate of tube

R_n	eigenfunctions
s	power law model index
T	temperature of fluid
T_0	inlet temperature
T_w	wall temperature
u_{av}	average velocity of fluid
x	axial coordinate of tube

Greek Symbols

α_n	eigenvalues
β	$\left(\frac{3s+1}{2s}\right)^{s+1} \frac{mU_{av}{}^{s+1}a^{1-s}}{k(T_0 - T_2)}$
δ	$\frac{qa^2}{k(T_0 - T_w)}$
$\Gamma_{n,m}$	integral defined by eq. (19)
η	$\frac{r}{a}$
θ	$\frac{T - T_w}{T_0 - T_w}$
θ_1	defined by eq. (10)
θ_2	defined by eq. (11)
θ_b	defined by eq. (20)
ξ	$\frac{2(s+1)x}{(3s+1)aPe}$

This work was performed under the auspices of the United States Energy Research and Development Administration under Contract No. EY-76-C-02-0016.

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Received June 21, 1978
Revised July 24, 1978